

# Hochschild homology and Gabber's Theorem

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## Introduction

About twenty-five O. Gabber proved his famous theorem [G] which claims, roughly speaking, that the singular support  $SS(\mathcal{F})$  of a  $D$ -module  $\mathcal{F}$  on a smooth algebraic manifold  $M$  is an *involutive* subvariety in the cotangent bundle  $T^*M$ . Involutive here means that the ideal sheaf  $\mathcal{J}$  defining  $SS(\mathcal{F})$  is closed with respect to the natural Poisson bracket on  $T^*M$  – that is,  $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$ .

This statement would have been very easy and straightforward, but there a complication:  $SS(\mathcal{F})$  here is taken with reduced scheme structure.

To understand the difficulty, let us recall the definition of  $SS(\mathcal{F})$ . One first finds a good increasing filtration on  $\mathcal{F}$  which in particular is compatible with the order filtration on the algebra  $D$  of differential operators. The associated graded quotient  $\text{gr } \mathcal{F}$  then becomes a module over  $\text{gr } D$ , and can be localized to a sheaf on  $T^*M = \text{Spec gr } D$ . By definition,  $SS(\mathcal{F})$  is the support of the sheaf  $\text{gr } \mathcal{F}$ . The annihilator  $\mathcal{J}' \subset \mathcal{O}_{T^*M}$  is easily seen to be involutive; however,  $\mathcal{J}'$  need not be a prime ideal sheaf, and the actual ideal sheaf  $\mathcal{J} \subset \mathcal{O}_{T^*M}$  defining  $SS(\mathcal{F}) \subset T^*M$  is the radical of  $\mathcal{J}'$ .

Thus the statement one has to prove is entirely algebraic, but highly non-trivial: there is no obvious reason why the radical of this particular involutive ideal should also be involutive (and it is certainly not true for all involutive ideals, e.g. the square of *any* ideal is involutive for trivial reasons). The general algebraic conjecture on involutivity of the singular support first appeared in [GQS]. In the years between [GQS] and [G], a lot of progress was made – following partial results in [GQS] itself and [Bon], a complete proof was given in [KKS]. But all these results used difficult analytic methods, such as microlocalization and pseudodifferential operators of infinite order. A different proof can be extracted from [KS], especially

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Chapter 11, but it also uses difficult analytic facts. The argument Gabber found was very beautiful and quite general, and it was purely algebraic – in fact, completely elementary in the sense that all the techniques used are contained in an undergraduate algebra course. However, the conceptual essence of the argument seems to be very hard to catch. Perhaps as the result of this, the standard textbooks on  $D$ -module theory such as [Bor] avoid it by using a trick of J. Bernstein (although some explanation is given in [CG]).

Recently considerable progress has been made in some areas of homological algebra, and we believe that one can now revisit Gabber’s Theorem and put it into a conceptual framework. From our point of view, the appropriate notions are those of Hochschild Homology and Cohomology for a small abelian category, and the associated deformation theory.

Unfortunately, at present these theories are under development, and far from being completed. The situation with Hochschild homology is better, since a very thorough treatment has been given by B. Keller [Ke] (nevertheless, there are fine points here as well, see Remark 2.3). The notion of Hochschild cohomology is in fact much simpler; however, developing a deformation theory for abelian categories based on Hochschild cohomology is a highly non-trivial matter, and it has been done only very recently by W. Lowen and M. Van den Bergh [LB1], [LB2], [L]. At the moment – possibly because of my lack of competence – it is not clear to me whether the theory constructed in these papers is strong enough. We note that the requirements for the Gabber’s Theorem are not very strong, but they are quite specific; when a general theory is being developed, they are likely to be omitted at first.

The present paper arose as an attempt to explain and generalize Gabber’s Theorem by first developing the Hochschild cohomology theory in an alternative way, which works in lesser generality than [Ke] and [LB1] but is better adapted to this particular problem. However, it soon became clear that a complete treatment would require some space. Therefore we have decided to split the text. While the theoretical paper [Ka] is being prepared, the present paper is intended to give the actual practical proof of the Gabber’s Theorem by using Hochschild homology, and to state clearly what general facts about Hochschild homology and cohomology one needs. Necessarily, to do this we have to quote some things without proof. Nevertheless, we have decided that doing this might be useful. If nothing else, this will show what statements should definitely be contained in a Hochschild homology package suitable for practical applications, and allow a reader who is prepared to accept on faith some general nonsense to understand how the

proof of Gabber’s Theorem really works.

The paper is organized as follows. In Section 1 we describe the general properties that one expects from the Hochschild homology and cohomology formalism – specifically, those properties that we need for the Gabber’s Theorem. So as not to leave the exposition completely without a foundation, in Section 2 we sketch a skeleton theory satisfying these requirements for sheaves on a (regular affine) scheme with supports in a closed subscheme. We leave two crucial compatibility results without proof (in a general theory, these should be more-or-less automatic). Finally, in Section 3 we show how this formal theory allows one to prove Gabber’s Theorem, and we indicate possible generalizations.

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## 1 General requirements.

Fix a field  $k$ , and let  $\mathcal{C}$  be a small abelian  $k$ -linear category. In our applications,  $\mathcal{C}$  will have finite global homological dimension. At least under this assumption, and possibly always, one associates to  $\mathcal{C}$  a pair of a homology and a cohomology theory, called Hochschild homology and Hochschild cohomology, which have some natural properties. Let us list those properties.

We start with Hochschild cohomology  $HH^\bullet(\mathcal{C})$  which is easier to understand. First of all, it is a graded-commutative algebra over  $k$ . Now, we note that the category  $\mathcal{F}un(\mathcal{C}, \mathcal{C})$  of right-exact functors from  $\mathcal{C}$  to itself is also an abelian  $k$ -linear category. The identity functor  $\text{Id}_{\mathcal{C}}$  is an object in  $\mathcal{F}un(\mathcal{C}, \mathcal{C})$ . Then the following must hold.

- There exists a natural algebra map  $\text{ev} : HH^\bullet(\mathcal{C}) \rightarrow \text{Ext}^\bullet(\text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}})$ , where the Ext-groups are computed in  $\mathcal{F}un(\mathcal{C}, \mathcal{C})$ .

In most practical cases, this natural map will be an isomorphism, and one can just take it as the definition of  $HH^*(\mathcal{C}, \mathcal{C})$ . However, in some situations things might be more complicated (for finiteness reasons). In any case, the map should always exist. In particular, for any object  $A \in \mathcal{C}$ , we have by restriction a natural *evaluation map*

$$\text{ev}_A : HH^*(\mathcal{C}) \rightarrow \text{Ext}_{\mathcal{C}}^*(A, A).$$

This map is compatible with the algebra structure on both sides.

The main interest in Hochschild cohomology is in its relation to deformation theory. Fortunately, for Gabber's Theorem we only need one-parameter deformations of order 1, so that there is no need to study obstructions (which would require, among other things, putting a Lie algebra structure on  $HH^*(\mathcal{C})$ ). We restrict our attention to the following situation. Let  $\mathcal{C}'$  be an abelian  $k$ -linear category equipped with a functorial endomorphism  $h : \text{Id}_{\mathcal{C}'} \rightarrow \text{Id}_{\mathcal{C}'}$  such that  $h^2 = 0$ , and let  $\mathcal{C} \subset \mathcal{C}'$  be the full abelian subcategory of objects  $A \in \mathcal{C}'$  annihilated by  $h$ . Denote by  $\iota : \mathcal{C} \rightarrow \mathcal{C}'$  the embedding functor. It is easy to see that it admits a right-adjoint functor  $\tau : \mathcal{C}' \rightarrow \mathcal{C}$  given by  $\tau(A) = A/h = \text{Coker}(h_A) \in \mathcal{C} \subset \mathcal{C}'$ .

**Definition 1.1.** The category  $\mathcal{C}'$  is a *first-order deformation* of the category  $\mathcal{C}$  if the left-adjoint functor  $\tau : \mathcal{C}' \rightarrow \mathcal{C}$  admits derived functors  $L^\bullet \tau : \mathcal{C}' \rightarrow \mathcal{C}$ , and for any  $k \geq 1$ , the composition  $L^k \tau \circ \iota$  is isomorphic to  $\text{Id}_{\mathcal{C}}$ . An object  $A \in \mathcal{C}'$  is said to be *flat* with respect to  $\mathcal{C} \subset \mathcal{C}'$  if  $L^k \tau(A) = 0$  for  $k \geq 1$ .

**Example 1.2.** If  $A$  is a  $k$ -algebra, and  $\mathcal{C}$  is the category of left  $A$ -modules, then the category  $\mathcal{C}'$  of left modules over  $A[h]/h^2 = A \otimes_k k[h]/h^2$  is obviously a first-order deformation of  $\mathcal{C}$ . The functor  $L^\bullet \tau : \mathcal{C}' \rightarrow \mathcal{C}$  is equal to  $\text{Tor}_{A[h]/h^2}^\bullet(A, -)$ . This generalizes immediately to a sheaf  $\mathcal{A}$  of  $k$ -algebras on a topological space  $X$ , and the category of sheaves of left  $\mathcal{A}$ -modules on  $X$ .

**Remark 1.3.** The isomorphism  $L^k \tau \circ \iota \cong \text{Id}_{\mathcal{C}}$  is a flatness condition on  $\mathcal{C}'$ . One can show that it is in fact sufficient to require it for  $k = 1$ . The same goes for flat objects.

By definition, derived functors  $L^k \tau$  of the right-exact functor  $\tau$  come equipped with an additional structure: they all fit into a complex  $L^\bullet \tau$ . In particular, truncating this complex to  $L^{\leq 1} \tau$ , we obtain by Yoneda a canonical class  $t_{\mathcal{C}'} \in \text{Ext}^2(\tau, L^1 \tau)$ . Composing this with the exact functor  $\iota : \mathcal{C} \rightarrow \mathcal{C}'$ , we get a class  $\iota(t_{\mathcal{C}'}) \in \text{Ext}^2(\tau \circ \iota, L^1 \tau \circ \iota) = \text{Ext}^2(\text{Id}_{\mathcal{C}}, \text{Id}_{\mathcal{C}})$ . It is this class that should parameterize deformations.

- To any first-order deformation  $\mathcal{C}'$  of the category  $\mathcal{C}$ , one associates a Hochschild cohomology class  $\Theta_{\mathcal{C}'} \in HH^2(\mathcal{C})$  such that

$$(1.1) \quad \text{ev}(\Theta_{\mathcal{C}'}) = \iota(t_{\mathcal{C}'}).$$

**Lemma 1.4.** *Assume given a first-order deformation  $\mathcal{C}' \supset \mathcal{C}$  and an object  $A \in \mathcal{C}$ . If there exists a flat object  $\tilde{A} \in \mathcal{C}'$  such that  $\tilde{A}/h \cong A$ , then  $\text{ev}_A(\Theta_{\mathcal{C}'}) = \iota(t_{\mathcal{C}'})_A = 0$ .*

*Proof.* To see explicitly the class  $\iota(t_{\mathcal{C}'})_A$ , we note that for any short exact sequence  $0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$  in  $\mathcal{C}'$ , we have a four-term exact sequence

$$(1.2) \quad 0 \longrightarrow \text{Coker } p \longrightarrow \tau(B) \longrightarrow \tau(C) \longrightarrow \tau(A) \longrightarrow 0,$$

where  $p$  is the natural map  $L^1\tau(C) \rightarrow L^1\tau(A)$ . It is this sequence that represents by Yoneda the class  $p \circ t_{\mathcal{C}',A} \in \text{Ext}^2(\tau(A), \text{Coker } p)$ . Now, we apply this to the natural exact sequence  $0 \rightarrow N \rightarrow \tilde{A} \rightarrow \iota(A) \rightarrow 0$ , where  $N$  the kernel of the surjective adjunction map  $\tilde{A} \rightarrow \iota(A)$ . Then since  $\tilde{A}$  is flat,  $p$  is an isomorphism, and the sequence (1.2) becomes

$$0 \longrightarrow L^1\tau(\tilde{A}) \cong A \longrightarrow \tau(N) \xrightarrow{0} A \longrightarrow A \longrightarrow 0,$$

which obviously represents 0.  $\square$

In fact, the converse to this statement should also be true; moreover, a full theory would show that up to an equivalence, first-order deformations  $\mathcal{C}'$  are classified by the corresponding deformation classes  $\Theta_{\mathcal{C}'} \in HH^2(\mathcal{C})$ . But we will not need this.

Let us now turn to homology. Hochschild homology of the category  $\mathcal{C}$  should be a graded module  $HH_\bullet(\mathcal{C})$  over the algebra  $HH^\bullet(\mathcal{C})$ . Fortunately, a very thorough theory of Hochschild (and cyclic) homology for abelian categories has been developed by B. Keller [Ke]; in particular, it is definitely known what are the groups  $HH_\bullet(\mathcal{C})$ . Keller's theory even works in a more general setting of exact categories, which allows to prove results such as invariance with respect to derived equivalences and so on. Unfortunately, Keller only works with homology, with no mention of cohomology. Therefore the  $HH^\bullet(\mathcal{C})$ -module structure on  $HH_\bullet(\mathcal{C})$  is not explicitly contained in his work. We will need this structure, and will in fact need more.

- For any object  $A \in \mathcal{C}$ , there exists a natural trace map

$$\text{tr}_A : \text{Ext}_\mathcal{C}^\bullet(A, A) \rightarrow HH_\bullet(\mathcal{C})$$

compatible with the  $HH^\bullet(\mathcal{C})$ -module structure on both sides.

The trace map also does not appear explicitly in [Ke], but one of its corollaries does. Namely, the trace map can be applied to the identity map  $\text{id}_A$ ; the result is the *Chern class*

$$\text{ch}_A = \text{tr}(\text{id}_A) \in HH_0(\mathcal{C}).$$

Keller's definition of the Chern class is slightly different, but its main property is the same.

- for any short exact sequence  $0 \rightarrow B \rightarrow C \rightarrow A \rightarrow 0$  in  $\mathcal{C}$  we have  $\text{ch}_C = \text{ch}_A + \text{ch}_B$ .

This is known as *devisseage*. Other properties of Hochschild homology are very prominent in [Ke]; among other things, they insure that  $HH_*(\mathcal{C})$  is what it should be for particular categories such as modules over an algebra or coherent sheaves on a scheme. Of these properties, we will need the following two.

- If  $\mathcal{C}_0 \subset \mathcal{C}$  is a thick (a.k.a. Serre) abelian subcategory, and  $\mathcal{C}/\mathcal{C}_0$  is the quotient abelian category, then there exists a natural long exact sequence

$$HH_*(\mathcal{C}_0) \longrightarrow HH_*(\mathcal{C}) \longrightarrow HH_*(\mathcal{C}/\mathcal{C}_0) \longrightarrow$$

(the so-called *excision property*). If  $\mathcal{C}$  is the category of finitely generated modules over a Noetherian algebra  $A$ , then  $HH_*(\mathcal{C}) \cong HH_*(A) = \text{Tor}^{A^{\text{opp}} \otimes A}(A, A)$ .

These properties are not used in the proof of the Gabber's Theorem directly; their only importance is that they allow to compute homology and cohomology of the relevant abelian category.

## 2 Computations for sheaves.

We will now sketch a direct – and therefore, somewhat ugly – construction of the Hochschild homology and cohomology theories in the particular case needed for the Gabber's Theorem; it will satisfy all the properties listed above. The category  $\mathcal{C}$  in question is the category  $\text{Shv}(X, Z)$  of sheaves on a smooth scheme  $X$  supported in a closed subset  $Z \subset X$ . We further assume that the scheme  $X$  is of finite type over a field  $k$ . We note that the case  $Z = X$ ,  $\text{Shv}(X, Z) = \text{Shv}(X)$  is more-or-less classic by now, see e.g. [W],

[S]. However, for the Gabber's Theorem, it is essential to be able fix the support.

Consider the product  $X \times_k X$ , and let  $\mathcal{O}_\Delta$  be the structure sheaf of the diagonal  $\Delta \in X \times X$ . Set  $HH^\bullet(\mathrm{Shv}(X)) = \mathrm{Ext}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ . We note that these Ext-groups can be computed by the Koszul resolution. In particular, they remain the same if computed in the category  $\mathrm{Shv}(X \times X, \Delta)$  of coherent sheaves on  $X \times X$  supported on the diagonal  $\Delta \subset X \times X$ . Every object  $\mathcal{F} \in \mathrm{Shv}(X \times X, \Delta)$  defines a right-exact functor  $K(\mathcal{F})$  from  $\mathrm{Shv}(X)$  to itself by

$$K(\mathcal{F})(\mathcal{E}) = p_{2*}(p_1^*\mathcal{E} \otimes \mathcal{F}),$$

where  $p_{1,2} : X \times X \rightarrow X$  are the natural projections. The correspondence  $\mathcal{F} \mapsto K(\mathcal{F})$  is an exact functor from  $\mathrm{Shv}(X \times X, \Delta)$  to  $\mathcal{F}\mathrm{un}(\mathrm{Shv}(X), \mathrm{Shv}(X))$  which sends  $\mathcal{O}_\Delta$  to the identity functor and induces therefore an algebra map

$$K : HH^\bullet(\mathrm{Shv}(X)) = \mathrm{Ext}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \rightarrow \mathrm{Ext}^\bullet(\mathrm{Id}_{\mathrm{Shv}(X)}, \mathrm{Id}_{\mathrm{Shv}(X)}),$$

which we take as our map  $\mathrm{ev}$ . If we restrict our attention to the subcategory  $\mathrm{Shv}(X, Z) \subset \mathrm{Shv}(X)$  supported in some closed  $Z \subset X$ , then all of the above goes through literally, with one change: one has to replace the product  $X \times X$  with its formal completion along  $Z \times Z \subset X \times X$ .

If  $X$  is affine, then the algebra  $HH^\bullet(\mathrm{Shv}(X))$  is easy to compute explicitly by using the Koszul resolution and the local-to-global spectral sequence, which in these assumptions degenerates: we have

$$HH^\bullet(\mathrm{Shv}(X)) = H^0(X, \Lambda^\bullet \mathcal{T}(X)),$$

where  $\Lambda^\bullet$  is the exterior algebra, and  $\mathcal{T}(X)$  is the tangent sheaf of the scheme  $X$  (since by assumption  $X$  is regular,  $\mathcal{T}(X)$  is a vector bundle). Passing from  $\mathrm{Shv}(X)$  to  $\mathrm{Shv}(X, Z)$  amount to taking completion with respect to the ideal  $\mathcal{J}_Z \subset \mathcal{O}_X$  defining  $Z \subset X$ . In particular, classes in  $HH^2(\mathrm{Shv}(X, Z))$  are just bivector fields on the formal completion of  $X$  along  $Z \subset X$ .

If  $X = \mathrm{Spec} A$  is affine, then the deformations of the category  $\mathrm{Shv}(X)$  that we will need come from deformations of the algebra  $A$  (we have already seen the trivial deformation in Example 1.2). Namely, by a first-order flat deformation of  $A$  we will understand an associative algebra  $A_h$  which is flat over  $k\langle h \rangle = k[h]/h^2$  and equipped with an isomorphism  $A_h/h \cong A$ . Then for any such  $A_h$ , the category of finitely-degenerated left  $A_h$ -modules obviously satisfies the assumptions of Definition 1.1. One checks easily that for any  $a, b \in A$  lifted to elements  $a', b' \in A_h$  we have

$$a'b' - b'a' = h(\Theta_{A_h} \lrcorner da \wedge ab)$$

for some bivector field  $\Theta_{A_h}$  on  $A$ , independent of  $a, b, a', b'$ . It is this bivector field  $\Theta_{A_h} \in HH^2(\mathrm{Shv}(X))$  that one associates to the deformation  $A_h$ . The first piece of general nonsense that we accept without proof in this paper is the following compatibility statement.

**Lemma 2.1.** *The class  $\Theta_{A_h}$  satisfies (1.1).*  $\square$

In other words, the (standard) definition of the deformation class  $\Theta_{A_h}$  agrees with the categorical definition. The proof is rather straightforward but tedious: one replaces the functor categories in the categorical definition by appropriate categories of bimodules, and computes the class  $\iota(t_{\mathcal{C}'})$  explicitly by using the Koszul complex. We feel that in the context of this paper, this is better left to the reader.

We now turn to the Hochschild homology. The groups  $HH_*(\mathrm{Shv}(X))$  are defined as

$$HH_*(\mathrm{Shv}(X)) = H^\bullet(X, L^\bullet \delta^* \mathcal{O}_\Delta),$$

where  $\delta : X \rightarrow X \times X$  is the diagonal embedding (in algebraic notation, we have  $L^\bullet \delta^* \mathcal{O}_\Delta = \mathrm{Tor}_{X \times X}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$ ). We note that if  $X$  is not affine,  $HH_*(\mathrm{Shv}(X))$  might be non-trivial both in positive and in negative degrees. When we consider  $\mathrm{Shv}(X, Z) \subset \mathrm{Shv}(X)$ , then the Excision property forces us to set

$$HH_*(\mathrm{Shv}(X, Z)) = H_Z^\bullet(L^\bullet \delta^* \mathcal{O}_\Delta),$$

where  $H_Z^\bullet$  denote cohomology with supports in  $Z$ . This might be non-trivial both in positive and in negative degrees even when  $X$  is affine. For regular  $X$ ,  $L^\bullet \delta^* \mathcal{O}_\Delta$  can be computed explicitly: the Hochschild-Kostant-Rosenberg Theorem [HKR] claims that

$$L^i \delta^* \mathcal{O}_\Delta \cong \Omega^i(X).$$

Thus the homology sheaves of the complex  $L^\bullet \delta^* \mathcal{O}_\Delta$  are sheaves  $\Omega^i(X)$  of  $i$ -forms on  $X$ , each placed in degree  $-i$ . If  $X$  is affine, then the natural  $HH^*(\mathrm{Shv}(X, Z))$ -structure on  $HH_*(\mathrm{Shv}(X, Z))$  is given by the usual contraction operation between a form and a polyvector field. We note, and this is important, that the complex  $L^\bullet \delta^* \mathcal{O}_\Delta$  on  $X$  is Serre self-dual.

The most difficult thing to define explicitly is the trace map. One approach is the following. Let  $\mathcal{E} \in \mathrm{Shv}(X)$  be a coherent sheaf on  $X$ . Then by adjunction, we have a natural class

$$a_{\mathcal{E}} \in \mathrm{Ext}^0(L^\bullet \delta^* \delta_* \mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}^0(\mathcal{E} \otimes L^\bullet \delta^* \mathcal{O}_\Delta, \mathcal{E}) \cong \mathrm{Ext}^0(\mathcal{E}, \mathcal{E} \otimes L^\bullet \delta^* \mathcal{O}_\Delta),$$

where the first isomorphism follows from the projection formula, and the second one uses the fact that  $L^\bullet \delta^* \mathcal{O}_\Delta$  is Serre self-dual. This class  $a_{\mathcal{E}}$ , while quite tautological, is not trivial; in characteristic 0, it incorporates the Atiyah class  $A \in \text{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \Omega^1(X))$  together with its exterior powers  $\Lambda^k(A) \in \text{Ext}^k(\mathcal{E}, \mathcal{E} \otimes \Omega^k(X))$  (see [M] for a detailed exposition). We note that while one usually considers the Atiyah class for vector bundles, there is no need to do so: the definition works just as well for every coherent sheaf. By transposition, for every coherent sheaf  $\mathcal{E}$  we obtain a class

$$a_{\mathcal{E}}^t \in \text{Ext}^0(\mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, \mathcal{E}), L^\bullet \delta^* \mathcal{O}_\Delta),$$

where  $\mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, \mathcal{E})$  is the complex which represents the local Ext-groups from  $\mathcal{E}$  to itself. Now, assume that  $\mathcal{E}$  is in fact supported in  $Z \subset X$ . Then so is the complex  $\mathcal{R}\mathcal{H}\text{om}(\mathcal{E}, \mathcal{E})$ . Therefore, by taking cohomology with supports in  $Z \subset X$ , the class  $a^t$  gives a map

$$H_Z^\bullet(\mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, \mathcal{E})) \cong \text{RHom}^\bullet(\mathcal{E}, \mathcal{E}) \rightarrow HH_\bullet(\text{Shv}(X, Z)).$$

This is our trace map  $\text{tr}_{\mathcal{E}}$ . We note that by its very construction, it is compatible with the natural  $HH^\bullet(\text{Shv}(X, Z))$ -module structures on both sides. Checking devissage for the associated Chern character  $\text{ch}_{\mathcal{E}} = \text{tr}_{\mathcal{E}}(\text{id}_{\mathcal{E}})$  is also straightforward. Indeed, for any short exact sequence

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

of sheaves on  $X$ , we can consider  $\mathcal{E}$  as a filtered sheaf (with a two-step filtration  $F_0 \mathcal{E} = \mathcal{E}_1$ ,  $F_1 \mathcal{E} = \mathcal{E}$ ). Then the algebra  $\mathcal{R}\mathcal{H}\text{om}_F^\bullet(\mathcal{E}, \mathcal{E})$  of filtered local Ext-groups is well-defined, and we have the commutative diagram

$$\begin{array}{ccc} \mathcal{R}\mathcal{H}\text{om}_F^\bullet(\mathcal{E}, \mathcal{E}) & \longrightarrow & \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}, \mathcal{E}) \\ \downarrow & & a_{\mathcal{E}}^t \downarrow \\ \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}_1) \oplus \mathcal{R}\mathcal{H}\text{om}^\bullet(\mathcal{E}_2) & \xrightarrow{a_{\mathcal{E}_1}^t \oplus a_{\mathcal{E}_2}^t} & L^\bullet \delta^* \mathcal{O}_\Delta \end{array}.$$

It remains to notice that  $\text{id}_{\mathcal{E}}$  comes from the class  $\text{id}_{\mathcal{E}}^F \in \text{Ext}_F^0(\mathcal{E}, \mathcal{E})$  which projects to the class  $\text{id}_{\mathcal{E}_1} \oplus \text{id}_{\mathcal{E}_2} \in \text{Ext}^0(\mathcal{E}_1, \mathcal{E}_1) \oplus \text{Ext}^0(\mathcal{E}_2, \mathcal{E}_2)$ .

The second general-nonsense compatibility result that we prefer to accept without any indication of proof concern an explicit computation of the Chern character in one simple case. Assume that the regular scheme  $X$  is affine, and moreover, assume that  $Z \subset X$  is also regular, of codimension  $l$ . Then the local cohomology module  $H_Z^p(X, \mathcal{O}_X)$  is trivial for  $p \neq l$ , and

$H_Z^l(X, \mathcal{O}_X)$  is well-known: this is the so-called (right)  $\delta$ -function  $D$ -module on  $X$  supported in  $Z$ . It has an increasing filtration whose associated graded quotient is isomorphic to

$$\text{gr}^\bullet H_Z^l(\mathcal{O}_X) \cong H^0(Z, \omega_{X,Z}^{-1} \otimes S^\bullet \mathcal{N}_{X,Z}),$$

where  $\mathcal{N}_{X,Z}$  is the normal bundle to  $Z$  in  $X$ ,  $S^\bullet$  stands for symmetric power, and  $\omega_{X,Z} = \Lambda^l(\mathcal{N}_{X,Z}^*)$  is the top exterior power of the conormal bundle  $\mathcal{N}_{X,Z}^*$ . In particular, we have a canonical embedding  $H^0(Z, \omega_{X,Z}^{-1}) \rightarrow H_Z^l(X, \mathcal{O}_X)$ . This gives canonical maps

$$H^0(Z, \omega_{X,Z}^{-1} \otimes \Omega^p(X)|_Z) \rightarrow HH_{p-l}(\text{Shv}(X, Z))$$

for any  $p$ ,  $0 \leq p \leq \dim X$ . But for  $p = l$ , the natural map  $\omega_{X,Z} \cong \Lambda^l \mathcal{N}_{X,Z}^* \rightarrow \Omega^l(X)|_Z$  gives a tautological class  $\tau_{X,Z}$  in  $H^0(Z, \omega_{X,Z}^{-1} \otimes \Omega^l(X)|_Z)$  and consequently, in  $HH_0(\text{Shv}(X, Z))$ .

**Lemma 2.2.** *In the assumptions above, the Chern character*

$$\text{ch}_{\mathcal{O}_Z} \in HH_0(\text{Shv}(X, Z))$$

*coincides with the tautological class  $\tau_{X,Z}$ .*

□

**Remark 2.3.** The real problem in this compatibility Lemma is to use a convenient construction of the embedding  $H^0(Z, \omega_{X,Z}^{-1}) \rightarrow H_Z^l(X, \mathcal{O}_X)$ . In practice, this embedding is usually obtained by using residues, as in [H]. For purely formal reasons such as Excision, any well-developed Hochschild homology formalism would contain this as an integral part, in the form of the Tate residue and its higher-dimensional generalizations. Then Lemma 2.2 would essentially be the definition of the class  $\tau_{X,Z}$ ; the only thing to prove would be  $\tau_{X,Z} \neq 0$ , and even this should be an easy exercise. Unfortunately, Tate residue is not part of the foundational paper [Ke] (this is exactly one of those applied results that are likely to be omitted at the first stages of development of the general theory).

### 3 Gabber's Theorem.

Having spent several pages, for better or for worse, on describing the general formalism of Hochschild Homology and Cohomology, we can now show how it helps to prove Gabber's Theorem. The proof is really short.

By the preliminary reductions done in [G], the theorem amounts to the following purely algebraic fact. Assume given a commutative algebra  $A$  over a field  $k$  of characteristic 0, a prime ideal  $\mathfrak{m} \subset A$ , and a finitely generated  $A$ -module  $M$  annihilated by a power of  $\mathfrak{m}$ . Assume that  $A$  and  $A/\mathfrak{m}$  are Noetherian and regular. Assume in addition that we are given a first-order one-parameter deformation  $A_h$  of the algebra  $A$ , and assume that there exists a left  $A_h$ -module  $M_h$  which is flat over  $k[h]/h^2$  and satisfies  $M_h/h \cong M$ . Define a bracket operation  $\{-, -\}$  on  $A$  by

$$a'b' - b'a' = h\{a, b\}$$

for any  $a, b \in A$ , where  $a', b' \in A_h$  are arbitrary liftings of the elements  $a, b$ . Since  $A_h/h \cong A$  is commutative, this does not depend on the choice of liftings; if  $A_h$  comes from a full one-parameter non-commutative deformation of  $A$ , then  $\{-, -\}$  is the associated Poisson bracket.

**Theorem 3.1.** *In the assumptions above, the ideal  $\mathfrak{m} \subset A$  is involutive,  $\{\mathfrak{m}, \mathfrak{m}\} \subset \mathfrak{m}$ .*

*Proof.* Let  $\Theta_{A_h} = \Theta$  be the bivector field on  $\text{Spec } A$  associated to the deformation  $A_h$ , so that  $\{a, b\} = \Theta \lrcorner (da \wedge db)$ . Then obviously  $\{\mathfrak{m}, \mathfrak{m}\} \subset \mathfrak{m}$  if and only if the  $\eta(\Theta) = 0$ , where  $\eta : \Lambda^2 \mathcal{T}(A) \rightarrow \Lambda^2 \mathcal{N}_{A, \mathfrak{m}}$  is the projection onto the second exterior power of the normal bundle  $\mathcal{N}_{A, \mathfrak{m}}$ . This is equivalent to

$$(3.1) \quad \Theta \lrcorner \omega = 0,$$

where  $\omega \in \Omega^\bullet(A)/\mathfrak{m}$  is the determinant of the conormal bundle  $\mathcal{N}_{A, \mathfrak{m}}^* \cong \mathfrak{m}/\mathfrak{m}^2$ . Take now  $X = \text{Spec } A$ , and let  $Z \subset X$  be defined by the ideal  $\mathfrak{m}$ . Then by Lemma 2.2, (3.1) is equivalent to

$$\Theta \cdot \text{ch}_{A/\mathfrak{m}} = 0 \in HH_{-2}(\text{Shv}(X, Z)).$$

To prove this, we note that  $HH_{-2}(\text{Shv}(X, Z))$  is a flat module over  $A/\mathfrak{m}$ , so that we can localize  $A$  at  $\mathfrak{m}$  and assume that  $\mathfrak{m} \subset A$  is a maximal ideal. Then on one hand,  $\Theta \cdot \text{ch}_M = \Theta \cdot \text{tr}_M(\text{id}_M) = \text{tr}_M(\Theta \cdot \text{id}_M) = \text{tr}_M(\text{ev}_M)$ , which is 0 by Lemma 1.4, and on the other hand, by devissage

$$\text{ch}_M = n \text{ch}_{A/\mathfrak{m}}$$

for some positive integer  $n$ .  $\square$

In a nutshell, the essential part of this proof is this: we want to reduce the claim for general  $M$  to the (simple) particular case when  $M$  is annihilated by  $\mathfrak{m}$ , not some power  $\mathfrak{m}^l$ . We do this by reducing the claim to statement about a Hochschild homology class, and then using devissage to replace  $M$  with the associated graded quotient  $\text{gr } M$  with respect to the  $\mathfrak{m}$ -adic filtration. Just as [GQS] and [G], our proof is essentially a “trace argument”; trace appears in the form of the Chern class map.

We note that the argument is extremely crude – localizing at  $\mathfrak{m}$ , the generic point of  $Z \subset X$ , we lose a lot of information. When one has a well-developed Hochschild homology formalism at one’s disposal, the claim could be considerably strengthened to obtain information about the Chern class of the module  $M$  (and the associated sheaf, if one no longer assumes that  $X$  is affine). In particular, it should be possible to obtain a general  $D$ -module version of the obvious fact that the vector bundle underlying a local system has trivial Chern classes. All this, however, remains a topic for future research.

One point in which our proof is seriously deficient in comparison to [G] is our assumption that  $X$  is regular. In  $D$ -module applications, this is a given; in fact, the  $D$ -module theory on singular varieties is developed in a different way, and Gabber’s Theorem for singular  $A$  would probably be useless for this. However, the Theorem itself holds in full generality – no assumptions of regularity are imposed in [G]. In our argument, they are not strictly needed, either. However, one has a feeling that a reasonable theory of Hochschild Homology can only be developed for categories of finite homological dimension. Thus for general  $A$ , our general nonsense statements are probably not true without serious modifications, and anything built on them becomes suspect.

## References

- [Bon] J.-M. Bony, *Quasi-analyticité et unicité du problème de Cauchy pour les solutions déquations aux dérivées partielles*, Séminaire Goulaouic-Schwartz 1970–1971: équations aux dérivées partielles et analyse fonctionnelle, Exp. No. 10, Centre de Math., École Polytech., Paris, 1971.
- [Bor] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic  $D$ -modules*, Perspectives in Mathematics, **2**, Academic Press, Inc., Boston, MA, 1987.
- [CG] N. Chriss and V. Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston, Inc., Boston, MA, 1997.
- [G] O. Gabber, *The integrability of the characteristic variety*, Amer. J. Math. **103** (1981), 445–468.

- [GQS] V. Guillemin, D. Quillen, and S. Sternberg, *The integrability of characteristics*, Comm. Pure and Appl. Math. **23** (1970), 39–77.
- [H] R. Hartshorne, *Residues and Duality*, Lecture Notes in Math. **20**, Springer-Verlag, Heidelberg, 1966.
- [HKR] G. Hochschild, B. Kostant, and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. AMS **102** (1962), 383–408.
- [Ka] D. Kaledin, *Hochschild homology and cohomology of abelian categories*, in preparation.
- [KKS] M. Kashiwara, T. Kawai, and M. Sato, *Microfunctions and pseudo-differential equations*, Lecture Notes in Math. **287** (1973), Springer-Verlag, 264–529.
- [KS] M. Kashiwara and P. Schapira, *Sheaves on Manifolds*, Springer-Verlag, 1994.
- [Ke] B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), 1–56.
- [LB1] W. Lowen and M. Van den Bergh, *Deformation theory of abelian categories*, math.CT/0405226.
- [LB2] W. Lowen and M. Van den Bergh, *Hochschild cohomology of abelian categories and ringed spaces*, math.KT/0405227.
- [L] W. Lowen, *Obstruction theory for objects in abelian and derived categories*, math.KT/0407019.
- [M] N. Markarian, Poincaré-Birkhoff-Witt isomorphism, Hochschild homology and Riemann-Roch theorem, preprint (2001), MPI 2001–52, available from Max Planck Institute’s web site.
- [S] R. Swan, *Hochschild cohomology of quasiprojective varieties*, J. of Pure and Appl. Algebra **110** (1996), 57–80.
- [W] C. Weibel, *Cyclic homology for schemes*, Proc. AMS **124** (1996), 1655–1662.

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